Mixed solutions of the Yang-Baxter equation and the associated knot and link invariants

A research problem

by

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Consider the Yang-Baxter equation in the form

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$$

where R is an $n^2 \times n^2$ matrix, R_{12} stands for $R \otimes I_n$ and $R_{23} = I_n \otimes R$.

In the paper: Michiel Hazewinkel, *Multiparameter quantum groups and multiparameter R-matrices*, Acta Appl. Math. **41** (1995), 57-98, all solutions of the Yang-Baxter equations were described that satisfy the additional condition

$$R^{a,b}_{c,d} = 0$$

unless $\{a, b\} = \{c, d\}$. The classical A-type solutions, i.e. the ones that define the quantum group deformations of the general linear and special linear group all satisfy this condition. It turns out that all solutions of the Yang-Baxter equations that satisfy the special additional condition mentioned above consist of (generalized) A-type blocks which can be nontrivially connected; in turn each block consists of cells. These cells are obtained by taking a scalar multiple of certain standard matrices; at this time it does not seem that considering cells of size larger than 1 brings anything much new. So below I will only talk about blocks in which each cell is of size 1×1.

To each block there is associated a number. A multiblock solution extends to a Yang-Baxter operator if and only if the block numbers are all equal. These solutions therefore define link invariants according to Turaev's formula (V G Turaev, *The Yang-Baxter equation and invariants of links*, Inv. Math **92** (1988), 527-553). This still leaves room for blocks to be nontrivially connected. Indeed taking two absolutely trivial blocks consisting of the unit matrix of size 1 and connecting them nontrivially by a parameter z one obtains a (new?) invariant that cannot distinguish between a nontrivial knot and the unknot; but it can count components in a link; it can see whether those components are nontrivially linked, and it contains information on how complicated the interlinking of components is; see the paper by M Hazewinkel cited above.

Now the classical (one block) solutions of A-type can be combined to yield the Jones polynomial; the classical solutions of D-type combine to yield the Kauffman polynomial. It is well known that the Jones polynomial and the Kauffman polynomial have rather different discriminatory powers as regards knots and links. It is possible to have solutions of the Yang-Baxter equations which consist of nontrivialy connected blocks of different types, e.g. A-type and D-type. (For more information on these kind of solutions see N van den Hijligenberg, Special solutions of the quantum Yang-Baxter equation, CWI preprint, 1997.)

Thus the research question arises what can be done in the way of constructing invariants by combining A-type and D-type blocks in a nontrivial way. The example described above (combining two totally trivial solutions) gives rise to some optimistic feelings of what might come out.